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Lipschitz Stability of Impulsive Systems of Differential Equations

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Notions of *Lipschitz stability* of the zero solution of impulsive systems of differential equations with fixed moments of impulse effect are introduced. Sufficient conditions for various types of uniform Lipschitz stability are obtained and the relations between these notions are investigated. The results obtained are used for the investigation of the uniform Lipschitz stability of the zero solution of linear impulsive systems of differential equations.

1. INTRODUCTION

The impulsive systems of differential equations are suitable mathematical models of numerous processes and phenomena in biology, physics, technology, etc. That is why in the recent years the mathematical theory of these systems has been developed by a great number of mathematicians (Bainov and Simeonov, 1989; Lakshmikantham *et al.*, 1989; Samoilenko and Perestyuk, 1987; Simeonov and Bainov, 1987). For a more detailed bibliography on this subject see Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), and Samoilenko and Perestyuk (1987).

In the present paper the notion of *Lipschitz stability* for impulsive systems of differential equations is introduced. For nonlinear systems of differential equations without impulses this notion was introduced by Dannan and Elaydi (1986).

For linear impulsive systems the notions of *uniform Lipschitz stability* and of *uniform stability by Lyapunov* are equivalent (Theorem 1). For nonlinear impulsive systems, however, the two notions are different (Example 4). In this paper it is proved that for impulsive systems of differential equations the notion of *uniform Lipschitz stability* [Definition 1(a)] is "between" the notions of *uniform stability* [Definition 1(e)] and *asymptotic stability in variations* [Definition 1(f)] (see Theorems 2 and 4).

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The results obtained are applied to the investigation of the uniform Lipschitz stability of the zero solution of linear impulsive systems (Examples 1 and 2).

2. PRELIMINARY NOTES AND DEFINITIONS

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $|x|$ be any norm of the vector $x \in \mathbb{R}^n$. Consider the impulsive system

$$\begin{aligned} \dot{x} &= f(t, x), \qquad t \neq t_k \\ \Delta x|_{t=t_k} &= I_k(x) \\ x(t_0+0) &= x_0 \end{aligned} \tag{1}$$

where $f: J \times \mathbb{R}^n \to \mathbb{R}^n$, $J = [t_0, \infty)$, $I_k: \mathbb{R}^n \to \mathbb{R}^n$, $0 \leq t_0 < t_1 < t_2 < \ldots$, and $\Delta x|_{t=t_k} = x(t_k+0) - x(t_k-0)$.

Impulsive systems of the form (1) were described in detail in Bainov and Simeonov (1989), Lakshmikantham *et al.* (1989), Samoilenko and Perestyuk (1987), and Simeonov and Bainov (1987).

Together with system (1), consider the impulsive variational systems

$$\begin{aligned} \dot{y} &= f_x(t, 0)y, \qquad t \neq t_k \\ \Delta y|_{t=t_k} &= I'_k(0)y \\ y(t_0+0) &= y_0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} \dot{z} &= f_x(t, x(t; t_0, x_0)z, \qquad t \neq t_k \\ \Delta z|_{t=t_k} &= I'_k(x(t_k; t_0, x_0)z \\ z(t_0+0) &= z_0 \end{aligned} \tag{3}$$

where $f_x = \partial f / \partial x$, $I'_k = \partial I_k / \partial x$, and $x(t; t_0, x_0)$ is the solution of system (1) satisfying the initial condition $x(t_0+0; t_0, x_0) = x_0$.

The fundamental matrix $\Phi(t, t_0, x_0)$ of system (3) is defined by the equality (Lakshmikantham *et al.*, 1989, Theorem 2.4.1)

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}(x(t; t_0, x_0)), \qquad t \neq t_k \tag{4}$$

and the fundamental matrix $\Psi(t, t_0)$ of system (2) by the equality

$$\Psi(t, t_0) = \Phi(t, t_0, 0) = \frac{\partial}{\partial x_0}(x(t; t_0, 0)), \qquad t \neq t_k \tag{5}$$

Henceforth we shall consider as well the linear impulsive system

$$\begin{aligned} \dot{x} &= A(t)x, \qquad t \neq t_k \\ \Delta x\big|_{t=t_k} &= B_k x \\ x(t_0+0) &= x_0 \end{aligned} \tag{6}$$

where $A(t)$ is an $n \times n$ matrix defined in J, and B_k, $k = 1, 2, \ldots$, are constant $n \times n$ matrices.

We shall not that if $U(t, s)$ is the fundamental matrix of the systems without impulses $\dot{x} = A(t)x$, then the fundamental matrix $W(t, s)$ of system (6) is defined by Bainov and Simeonov, 1989; Lakshmikantham *et al.*, 1989; Samoilenko and Perestyuk, 1987; Simeonov and Bainov, 1987)

$$W(t, s) = \begin{cases} U(t, s), & t_{k-1} < s \leq t \leq t_k \\ U(t, t_k)(E + B_k)U(t_k, s), & t_{k-1} < s \leq t_k < t \leq t_{k+1} \\ U(t, t_{k+i})\left[\prod_{j=i}^{1} (E + B_{k+j})U(t_{k+j}, t_{k+j-1})\right] & \\ \quad \times (E + B_k)U(t_k, s), & t_{k-1} < s \leq t_k < t_{k+i} < t \leq t_{k+i+1} \end{cases} \tag{7}$$

where E is the unit $n \times n$ matrix.

Straightforward calculations show that

$$\frac{\partial}{\partial t} W(t, s) = A(t) W(t, s), \qquad s < t, \quad t \neq t_k, \quad k = 1, 2, \ldots$$

$$W(s, s) = E$$

$$W(t_k + 0, s) = (E + B_k) W(t_k, s), \qquad s < t_k, \quad k = 1, 2, \ldots$$

$$W(t, s) W(s, t_0) = W(t, t_0), \qquad t_0 < s < t_0$$

We shall say that conditions (A) are met if the following conditions hold:

A1. $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

A2. The function $f\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and has a continuous partial derivative f_x in $(t_{k-1}, t_k] \times \mathbb{R}^n$, $k = 1, 2, \ldots$, and $f(t, 0) \equiv 0$.

A3. For any $x \in \mathbb{R}^n$ and any $k = 1, 2, \ldots$, the functions f and f_x have finite limits as $(t, y) \to (t_k, x)$, $t > t_k$.

A4. The functions $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots$, are continuously differentiable in $\mathbb{R}^n$ and $I_k(0) = 0$, $k = 1, 2, \ldots$.

A5. The solution $x(t; t_0, x_0)$ of system (1) which satisfies the initial condition $x(t_0 + 0; t_0, x_0) = x_0$ is defined in the interval (t_0, ∞).

We shall say that condition (B) is met if the following condition holds:

B. The matrix $A(t)$ is piecewise continuous in J with points of discontinuity of the first kind $t = t_k$, $k = 1, 2, \ldots$, at which it is continuous from the left.

We shall introduce definitions of various types of stability of the zero solution of system (1) which are analogous to the definitions given in Dannan and Elaydi (1986).

Definition 1. The zero solution of system (1) is said to be:

(a) *Uniformly Lipschitz stable* if

$$(\exists M>0)(\exists \delta>0)(\forall x_0 \in \mathbb{R}^n,\ |x|<\delta)(\forall t>t_0\geq 0):$$
$$|x(t;\, t_0 x_0)| \leq M|x_0|$$

(b) *Globally uniformly Lipschitz stable* if

$$(\exists M>0)(\forall x_0 \in \mathbb{R}^n)(\forall t>t_0\geq 0):\ |x(t;\, t_0,\, x_0)| \leq M|x_0|$$

(c) *Uniformly Lipschitz stable in variations* if

$$(\exists M>0)(\exists \delta>0)(\forall x_0 \in \mathbb{R}^n,\ |x_0|<\delta)(\forall t>t_0\geq 0):$$
$$\|\Phi(t,\, t_0,\, x_0)\| \leq M$$

(d) *Globally uniformly Lipschitz stable in variations* if

$$(\exists M>0)(\forall x_0 \in \mathbb{R}^n)(\forall t>t_0\geq 0):\ \|\Phi(t,\, t_0,\, x_0)\| \leq M$$

(e) *Uniformly stable* if

$$(\forall \varepsilon>0)(\exists \delta=\delta(\varepsilon)>0)(\forall t_0\geq 0)(\forall x_0 \in \mathbb{R}^n,\ |x_0|<\delta)(\forall t>t_0):$$
$$|x(t;\, t_0,\, x_0)|<\varepsilon$$

(f) *Asymptotically stable in variations* if

$$(\exists M>0)(\forall t>t_0\geq 0):$$
$$\int_{t_0}^{t} \|\Psi(t,s)\|\, ds \leq M \quad \text{and} \quad \sum_{t_0<t_k<t} \|\Psi(t,\, t_k+0)\| \leq M$$

In the further considerations we shall use the following notation:

(a) $\|A\| = \sup_{|x|\leq 1} |Ax|$, where A is an arbitrary $n \times n$ matrix.
(b) $S(\rho) = \{x \in \mathbb{R}^n : |x|<\rho\}$, $\rho>0$.
(c) $\mathscr{K} = \{a \in C[\mathbb{R}_+, \mathbb{R}_+]$: a is strictly increasing in $\mathbb{R}_+$ and $a(0)=0\}$.

3. MAIN RESULTS

Theorem 1. Let condition (B) hold. Then the following assertions are equivalent:

(i) The zero solution of (6) is globally uniformly Lipschitz stable in variations.
(ii) The zero solution of (6) is uniformly Lipschitz stable in variations.

(iii) The zero solution of (6) is globally uniformly Lipschitz stable.
(iv) The zero solution of (6) is uniformly Lipschitz stable.
(v) The zero solution of (6) is uniformly stable.

Proof. (i)⇒(ii). This follows immediately from Definitions 1(c) and 1(d).

(ii)⇒(iii). Let the zero solution of (4) be uniformly Lipschitz stable in variations. Then there exist constants $M>0$ and $\delta>0$ such that

$$\|W(t, t_0)\| \leq M \qquad \text{for} \quad t>t_0\geq 0, \quad |x_0|<\delta \tag{8}$$

Since the fundamental matrix W of (4) does not depend on x_0, then (8) holds for any $x_0 \in \mathbb{R}^n$. Consequently,

$$|x(t; t_0, x_0)| = |W(t, t_0)x_0| \leq \|W(t, t_0)\| \, |x_0| \leq M|x_0|$$

for $t>t_0\geq 0$ and $x_0 \in \mathbb{R}^n$, i.e., (iii), holds.

(iii)⇒(iv). This follows immediately from Definitions 1(a) and 1(b).

(iv)⇒(v). Let (iv) hold. Then there exist constants $M>0$ and $\delta_1>0$ such that $|x(t; t_0, x_0)| \leq M|x_0|$ whenever $|x_0|<\delta_1$ and $t>t_0\geq 0$.

Let $\varepsilon>0$ be given and let $\delta=\delta(\varepsilon)=\min(\delta_1, \varepsilon/M)$. Then for $|x_0|<\delta$ and $t>t_0\geq 0$ we have $|x(t; t_0, x_0)| \leq M|x_0| \leq M\delta<\varepsilon$, which shows that the zero solution of (4) is uniformly stable.

(v)⇒(i) From the uniform stability of the zero solution of (4) it follows that $\|W(t, t_0\| \leq M$ for $t>t_0\geq 0$ (Samoilenko and Perestyuk, 1987, Theorem 8.1). Hence (i) holds. ■

Remark 1. We shall note that Theorem 1 still holds for any system of the form (1) whose fundamental matrix W does not depend on x_0. Such is, for instance, the linear nonhomogeneous impulsive system

$$\dot{x} = A(t)x + h(t), \qquad t \neq t_k$$

$$\Delta x|_{t=t_k} = B_k x + b_k$$

$$x(t_0+0) = x_0$$

(see Simeonov and Bainov, 1987, p. 266).

Theorem 2. Let conditions (A) hold and let the zero solution of (1) be uniformly Lipschitz stable. Then the zero solution of (1) is uniformly stable.

Proof. From the uniform Lipschitz stability of the zero solution of (1) it follows that there exist constants $M>0$ and $\delta_1>0$ such that $|x(t; t_0, x_0)| \leq M|x_0|$ for $|x_0|<\delta_1$ and $t>t_0\geq 0$.

Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) = \min(\delta_1, \varepsilon/M)$. Then for $|x_0| < \delta$ and $t > t_0 \geq 0$ the inequalities $|x(t; t_0, x_0)| \leq M|x_0| < M\delta < \varepsilon$ are valid. Hence the zero solution of (1) is uniformly stable. ■

Theorem 3. Let condition (B) hold and let W be the fundamental matrix of system (6). Moreover, let piecewise continuous functions $k, h: J \to (0, \infty)$ exist with points of discontinuity $t = t_k$, $k = 1, 2, \ldots$, at which they are continuous from the left and such that

$$\int_{t_0}^{t} h(s)\| W(t, s)\| \, ds \leq k(t), \qquad t > t_0 \geq 0, \quad t \neq t_k, \quad k = 1, 2, \ldots \tag{9}$$

$$k(t) \exp\left(-\int_{t^*}^{t} \frac{h(s)}{k(s)} \, ds\right) \leq N, \qquad t > t^* > t_0 \geq 0, \qquad t \neq t_k \tag{10}$$

where $0 < N = \text{const}$.

Then the zero solution of (6) is uniformly Lipschitz continuous.

Proof. Let $b(t) = 1/\| W(t, t_0)\|$, $t > t_0 \geq 0$. Then

$$\left[\int_{t_0}^{t} h(s)b(s) \, ds\right] W(t, t_0) = \int_{t_0}^{t} W(t, s) W(s, t_0) h(s) b(s) \, ds$$

Hence

$$\frac{1}{b(t)} \int_{t_0}^{t} h(s)b(s) \, ds \leq \int_{t_0}^{t} \| W(t, s)\| h(s) \, ds \leq k(t) \tag{11}$$

where we have used (9).

Set $B(t) = \int_{t_0}^{t} h(s)b(s) \, ds$. Then $B(t)$ is continuous for $t \geq t_0 \geq 0$ and $B'(t) = h(t)b(t)$ for $t \neq t_k$, $k = 1, 2, \ldots$. Moreover, from (11) it follows that $h(t)B(t) \leq k(t)B'(t)$, i.e.,

$$B'(t) \geq \frac{h(t)}{k(t)} B(t), \qquad t > t_0 \geq 0, \qquad t \neq t_k, \quad k = 1, 2, \ldots \tag{12}$$

Multiply both sides of (12) by $\exp\{-\int_{t^*}^{t} [h(s)/k(s)] \, ds\}$ for some $t^* > t_0$ and obtain

$$\frac{d}{dt}\left[B(t) \exp\left(-\int_{t^*}^{t} \frac{h(s)}{k(s)} \, ds\right)\right] \geq 0, \qquad t \geq t^*, \quad t \neq t_k, \quad k = 1, 2, \ldots$$

Hence

$$B(t) \exp\left\{-\int_{t^*}^{t} [h(s)/k(s)] \, ds\right\} \geq B(t^*)$$

from which, using (11), we obtain

$$k(t)b(t) \geq B(t) \geq B(t^*) \exp\left\{\int_{t^*}^{t} [h(s)/k(s)]\, ds\right\}$$

Then

$$\|W(t, t_0)\| = \frac{1}{b(t)} \leq \frac{k(t)}{B(t^*)} \exp\left(-\int_{t^*}^{t} \frac{h(s)}{k(s)}\, ds\right)$$

Let $M = N/B(t^*)$. Then, applying (10), we obtain that $\|W(t, t_0)\| \leq M$ for $t > t_0 \geq 0$, which means that the zero solution of (6) is globally uniformly Lipschitz stable in variations. From Theorem 1 it follows that the zero solution of (6) is uniformly Lipschitz stable. ■

Corollary 1. Let conditions (A) hold and let the zero solution of (1) be asymptotically stable in variations. Then the zero solution of (2) is uniformly Lipschitz stable.

Proof. From the asymptotic stability in variations of the zero solution of (1) it follows that inequalities (9) and (10) of Theorem 3 hold with $h(t) \equiv 1$, $k(t) \equiv M$, $W(t, t_0) = \Psi(t, t_0)$, and $N = M$. Then the assertion of Corollary 1 follows from Theorem 3. ■

Corollary 2. Let condition (B) hold and let the fundamental matrix W of the linear system (6) satisfy the inequality

$$\int_{t_0}^{t} \|W(t, s)\|\, ds \leq t \qquad \text{for} \quad t > t_0 \geq 0$$

Then there exists a constant $M > 0$ such that $\|W(t, t_0)\| \leq M$ for $t > t_0 \geq 0$.

Theorem 4. Let conditions (A) hold and let the zero solution of (1) be asymptotically stable in variations. Then the zero solution of (1) is uniformly Lipschitz stable.

Proof. Let Ψ be the fundamental matrix of (2). From Corollary 1 it follows that $\|\Psi(t, t_0)\| \leq K_1$ for $t > t_0 \geq 0$, where $0 < K_1 = \text{const}$. From the condition of Theorem 4 it follows that

$$\int_{t_0}^{t} \|\Psi(t, s)\|\, ds \leq K_2 \qquad \text{and} \qquad \sum_{t_0 < t_k < t} \|\Psi(t, t_k + 0)\| \leq K_2$$

for $t > t_0 \geq 0$, where $0 < K_2 = \text{const}$.

Let $K = \max(K_1, K_2)$. Since $f(t, 0) \equiv 0$ and $I_k(0) = 0$, $k = 1, 2, \ldots$, then for $\varepsilon < 1/2K$ there exists $\delta > 0$ such that $f(t, x) = f_x(t, 0)x + h(t, x)$ and $I_k(x) = I'_k(0)x + h_k(x)$ for $|x| < \delta$, where $|h(t, x)| < \varepsilon|x|$ and $|h_k(x)| < \varepsilon|x|$.

Applying the variation of parameter formula (Simeonov and Bainov, 1987, p. 266), we obtain

$$
\begin{aligned}
|x(t;\, t_0, x_0)| &= \Big|\Psi(t, t_0+0)x_0 + \int_{t_0}^{t} \Psi(t, s)h(s, x(s;\, t_0, x_0))\, ds \\
&\quad + \sum_{t_0<t_k<t} \Psi(t, t_k+0)h_k(x(t_k;\, t_0, x_0))\Big| \\
&\le \|\Psi(t, t_0+0)\|\, |x_0| + \int_{t_0}^{t} \|\Psi(t, s)\|\, |h(s, x(s;\, t_0, x_0))|\, ds \\
&\quad + \sum_{t_0<t_k<t} \|\Psi(t, t_k+0)\|\, |h_k(x(t_k;\, t_0, x_0))| \\
&\le K|x_0| + \varepsilon \int_{t_0}^{t} \|\Psi(t, s)\|\, |x(s;\, t_0, x_0)|\, ds \\
&\quad + \varepsilon \sum_{t_0<t_k<t} \|\Psi(t, t_k+0)\|\, |x(t_k;\, t_0, x_0)| \\
&\le K|x_0| + 2\varepsilon K \sup_{t_0<s\le t} |x(s;\, t_0, x_0)|
\end{aligned}
$$

Hence

$$
|x(t;\, t_0, x_0)| \le \frac{K}{1-2\varepsilon K}|x_0| = M|x_0|, \qquad t > t_0 \ge 0
$$

This completes the proof of Theorem 4. ∎

Theorem 5. Let the following conditions be fulfilled:

1. Conditions (A) hold.
2. $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, 0) \equiv 0$.
3. For $(t, x) \in J \times S(\rho)$ and for any $h > 0$ small enough the following inequality is valid:

$$
|x + hf(t, x)| \le |x| + hg(t, |x|) + \varepsilon(h) \tag{13}
$$

where $\varepsilon(h)/h \to 0$ as $h \to 0$.

4. For $x \in S(\rho)$ and for any $k = 1, 2, \ldots$ the inequalities

$$
|x + I_k(x)| \le G_k(|x|) \tag{14}
$$

are valid, where $G_k\colon [0, \rho_0) \to [0, \rho)$ and $G_k \in \mathcal{K}$.

5. The zero solution of the scalar impulsive differential equation

$$
\begin{aligned}
\dot{u} &= g(t, u), \qquad t \ne t_k \\
u(t_k+0) &= G_k(u(t_k)) \\
u(t_0+0) &= u_0 \ge 0
\end{aligned} \tag{15}
$$

is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Then the zero solution of system (1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Proof. Let $\rho^* = \min(\rho, \rho_0)$. From condition 5 of Theorem 5 it follows that there exist constants $M > 0$ and $\delta > 0$ $(M\delta < \rho^*)$ such that

$$u(t;\, t_0, u_0) \leq Mu_0 \qquad \text{for} \quad 0 \leq u_0 < \delta, \qquad t > t_0 \geq 0 \tag{16}$$

where $u(t;\, t_0, u_0)$ is any solution of (15) for which $u(t_0+0;\, t_0, u_0) = u_0$.

We shall prove that $|x(t;\, t_0, x_0)| \leq M|x_0|$ for $|x_0| < \delta$ and $t > t_0 \geq 0$. Suppose that this is not true. Then there exists a solution $x(t) = x(t;\, t_0, x_0)$ of (1), $|x_0| < \delta$, and $t^* \in (t_k, t_{k+1}]$ for some positive integer k such that

$$|x(t^*)| > M|x_0| \qquad \text{and} \qquad |x(t)| \leq M|x_0| \qquad \text{for} \quad t_0 < t \leq t_k$$

From (14) it follows that

$$\begin{aligned} |x(t_k+0)| &= |x(t_k) + I_k(x(t_k))| \leq G_k(|x(t_k)|) \\ &\leq G_k(M|x_0|) < G_k(M\delta) \leq G_k(\rho^*) \leq \rho \end{aligned}$$

Hence there exists t^0, $t_k < t^0 \leq t^*$, such that

$$M|x_0| < |x(t^0)| < \rho \qquad \text{and} \qquad |x(t)| < \rho, \qquad t_0 < t \leq t^0 \tag{17}$$

Set $m(t) = |x(t)|$ and $u_0 = |x_0|$. From (13) it follows that for $t \in (t_0, t^0]$, $t \neq t_j$, $j = 1, 2, \ldots, k$, the following inequalities hold:

$$\begin{aligned} m'(t) &= \lim_{h\to 0}(1/h)[|x(t+h)| - |x(t)|] \\ &\leq \lim_{h\to 0}(1/h)[|x(t+h)| + hg(t, |x(t)|) + \varepsilon(h) - |x(t) + hf(t, x(t))|] \\ &\leq g(t, |x(t)|) + \lim_{h\to 0} \varepsilon(h)/h + \lim_{h\to 0} |(1/h)[x(t+h) - x(t)] - f(t, x(t))| \\ &= g(t, |x(t)|) = g(t, m(t)) \end{aligned}$$

From (14) we obtain that for $j = 1, 2, \ldots, k$ the inequalities

$$m(t_j+0) = |x(t_j+0)| = |x(t_j) + I_j(x(t_j))| \leq G_j(|x(t_j)|)$$

hold, hence

$$m(t_j+0) \leq G_j(|m(t_j)|), \qquad j = 1, 2, \ldots, k$$

Moreover,

$$m(t_0+0) = |x(t_0+0)| = |x_0| = u_0$$

Applying the comparison theorem (Lakshmikantham *et al.*, 1989, Theorem 1.4.3), we obtain

$$|x(t)| = m(t) \leq u(t;\, t_0, u_0), \qquad t_0 < t \leq t^0 \tag{18}$$

From (16)-(18) it follows that

$$M|x_0| < |x(t^0)| = m(t^0) \leq u(t^0;\, t_0, u_0) \leq Mu_0 = M|x_0|$$

The contradiction obtained shows that $|x(t;\, t_0, x_0)| \leq M|x_0|$ for $|x_0| < \delta$ and $t > t_0 \geq 0$.

Theorem 5 is proved. ■

Theorem 6. Let the following conditions be fulfilled:

1. Conditions 1, 2, 4, and 5 of Theorem 5 hold.
2. For $(t, x) \in J \times S(\rho)$ the inequality

$$[x, f(t, x)]_+ \leq g(t, |x|) \tag{19}$$

is valid, where

$$[x, y]_+ = \limsup_{h \to 0^+} (1/h)[|x + hy| - |x|], \qquad x, y \in \mathbb{R}^n$$

Then the zero solution of system (1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

Proof. Let $\rho^* = \min(\rho, \rho_0)$. From condition 5 of Theorem 5 it follows that there exist constants $M > 0$ and $\delta > 0$ $(M\delta < \rho^*)$ such that

$$u(t;\, t_0, u_0) \leq Mu_0 \qquad \text{for} \quad 0 \leq u_0 < \delta \quad \text{and} \quad t > t_0 \geq 0$$

We shall prove that $|x(t;\, t_0, x_0)| \leq M|x_0|$ for $|x_0| < \delta$ and $t > t_0 \geq 0$. Suppose that this is not true and as in the proof of Theorem 5 we find a solution $x(t) = x(t;\, t_0, x_0)$, $|x_0| < \delta$, of system (1) and $t^0 \in (t_k, t_{k+1}]$ for some positive integer k such that

$$M|x_0| < |x(t^0)| < \rho \qquad \text{and} \qquad |x(t)| < \rho, \qquad t_0 < t \leq t^0$$

Set $m(t) = |x(t)|$ and $u_0 = |x_0|$ and using (19) obtain that for $t \in (t_0, t^0]$, $t \neq t_j$, $j = 1, 2, \ldots, k$, the following inequalities hold:

$$\begin{aligned} D^+ m(t) &= \limsup_{h \to 0^+} (1/h)[m(t+h) - m(t)] \\ &= \limsup_{h \to 0^+} (1/h)[|x(t+h)| - |x(t)|] \\ &\leq \limsup_{h \to 0^+} |(1/h)[x(t+h) - x(t)] - f(t, x(t))| \\ &\quad + \limsup_{h \to 0^+} (1/h)[|x(t) + hf(t, x(t))| - |x(t)|] \\ &= [x(t), f(t, x(t))]_+ \leq g(t, m(t)) \end{aligned}$$

Later the proof of Theorem 6 is completed as the proof of Theorem 5. ■

Theorem 7. Let the following conditions be fulfilled:

1. Conditions (A) hold.

2. The function g is continuous, nondecreasing, positive, and submultiplicative in $I=(0,\infty)$ and

$$g(\lambda u)\geq \mu(\lambda)g(u) \qquad \text{for} \quad \lambda>0, u>0$$

where $\mu(\lambda)>0$ for $\lambda>0$.

3. For $(t,x)\in J\times\mathbb{R}^n$ the following inequality holds:

$$|f(t,x)|\leq m(t)g(|x|)$$

where the function $m(t)$ is continuous and nonnegative in J.

4. For $x\in\mathbb{R}^n$ and for any $k=1,2,\ldots$ the inequalities

$$|I_k(x)|\leq \beta_k|x|$$

hold, where β_k, $k=1,2,\ldots$, are nonnegative constants.

5. We have

$$G^{-1}\left[G\left(\prod_{\theta<t_k<\infty}(1+\beta_k)\right)+\frac{g(|x_0|)}{|x_0|}\int_\theta^\infty \prod_{s<t_k<\infty}\frac{1+\beta_k}{\mu(1+\beta_k)}\,m(s)\,ds\right]<\infty$$

for any $x_0\in\mathbb{R}^n$ and any $\theta\geq t_0\geq 0$, where

$$G(u)=\int_a^u\frac{ds}{g(s)}, \qquad u\geq a>0$$

and G^{-1} is the inverse to G.

6. $G(\infty)=\infty$.

Then the zero solution of system (1) is globally uniformly Lipschitz stable.

Proof. Since

$$x(t;t_0,x_0)=x_0+\int_{t_0}^t f(s,x(s;t_0,x_0))\,ds+\sum_{t_0<t_k<t} I_k(x(t_k;t_0,x_0))$$

then, using conditions 2–4 of Theorem 7, we obtain the inequalities

$$\begin{aligned}
&\frac{|x(t;t_0,x_0)|}{|x_0|}\\
&\quad\leq 1+\int_{t_0}^t\frac{m(s)}{|x_0|}\,g\left(|x_0|\frac{|x(s;t_0,x_0)|}{|x_0|}\right)ds+\sum_{t_0<t_k<t}\beta_k\frac{|x(t_k;t_0,x_0)|}{|x_0|}\\
&\quad\leq 1+\int_{t_0}^t\frac{g(|x_0|)}{|x_0|}\,m(s)g\left(\frac{|x(s;t_0,x_0)|}{|x_0|}\right)ds+\sum_{t_0<t_k<t}\beta_k\frac{|x(t_k;t_0,x_0)|}{|x_0|}
\end{aligned}$$

Then from the impulsive integral inequality (Lakshmikantham *et al.*, 1989, Theorem 1.5.5) it follows that

$$|x(t;\, t_0, x_0)| \le x_0 |G^{-1}\biggl[G\biggl(\prod_{t_0<t_k<t}(1+\beta_k)\biggr) + \frac{g(|x_0|)}{|x_0|}\int_{t_0}^{t}\prod_{s<t_k<t}\frac{1+\beta_k}{\mu(1+\beta_k)}\,m(s)\,ds\biggr]$$

From condition 5 of Theorem 7 it follows that $|x(t;\, t_0, x_0)| \le M|x_0|$ for any $x_0 \in \mathbb{R}^n$ and for $t > t_0 \ge 0$.

This completes the proof of Theorem 7. ■

Theorem 8. Let the following conditions be fulfilled:

1. Conditions (A) hold.
2. The function g is continuous, nondecreasing, positive, and submultiplicative in $I = (0, \infty)$.
3. Conditions 3 and 4 of Theorem 7 hold.
4. For any $k = 1, 2, \ldots,$ $t \in (t_k, t_{k+1}]$, and $x_0 \in \mathbb{R}^n$ the following inequalities hold:

$$G_k^{-1}\biggl(\frac{g(|x_0|)}{|x_0|}\int_{t_k}^{t} m(s)\,ds\biggr) \le M$$

where $0 < M = \text{const}$,

$$G_k(u) = \int_{c_k}^{u}\frac{ds}{g(s)}, \qquad c_k = (1+\beta_k)G_{k-1}^{-1}\biggl(\int_{t_{k-1}}^{t_k} m(s)\,ds\biggr), \qquad k = 1, 2, \ldots$$

$$G_0(u) = \int_{c}^{u}\frac{ds}{g(s)}, \qquad u \ge c > 0$$

and G_k^{-1} is the inverse to G_k.

5. $G_k(\infty) = \infty$, $k = 0, 1, 2, \ldots.$

Then the zero solution of system (1) is globally uniformly Lipschitz stable.

Proof. As in the proof of Theorem 7, we obtain the inequality

$$\frac{|x(t;\, t_0, x_0)|}{|x_0|} \le 1 + \int_{t_0}^{t}\frac{g(|x_0|)}{|x_0|}\,m(s)\,g\biggl(\frac{|x(s;\, t_0, x_0)|}{|x_0|}\biggr)\,ds + \sum_{t_0<t_k<t}\mu_k\frac{|x(t_k;\, t_0, x_0)|}{|x_0|}, \qquad t > t_0 \ge 0$$

Applying the impulsive integral inequality (Samoilenko and Perestyuk, 1987, Lemma 16.1), we obtain

$$|x(t;\, t_0, x_0)| \le |x_0|\,G_k^{-1}\biggl(\frac{g(|x_0|)}{|x_0|}\int_{t_0}^{t} m(s)\,ds\biggr), \qquad t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots$$

Then from condition 4 of Theorem 8 it follows that $|x(t;\, t_0, x_0)| \le M|x_0|$ for $x_0 \in \mathbb{R}^n$ and $t > t_0 \ge 0$.

Theorem 8 is proved. ■

Theorem 9. Let conditions (A) hold and, moreover, for $\theta \ge t_0 \ge 0$ and $|x_0| < \delta$ let

$$\int_\theta^\infty \Lambda(t, t_0, x_0)\, dt < \infty \qquad \text{and} \qquad \prod_{\theta < t_k < \infty} \Lambda_k < \infty \tag{20}$$

where $\Lambda(t, t_0, x_0)$ is the greatest eigenvalue of the matrix

$$\tfrac{1}{2}[f_x(t, x(t;\, t_0, x_0)) + f_x^T(t, x(t;\, t_0, x_0))]$$

and Λ_k, $k = 1, 2, \ldots$, are the greatest eigenvalues of the matrices $[E + I_k'(0)][E + I_k'(0)]^T$.

Then the zero solution of system (1) is uniformly Lipschitz stable.

Proof. From Samoilenko and Perestyuk (1987, Theorem 9.1) it follows that

$$\|\Phi(t, t_0, x_0)\| \le \left[\prod_{t_0 < t_k < t} \Lambda_k\right] \exp\left[\int_{t_0}^t \Lambda(s, t_0, x_0)\, ds\right]$$

where

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}(x(t;\, t_0, x_0)), \qquad t \ne t_k, \quad k = 1, 2, \ldots$$

is the fundamental matrix of system (3).

Then from conditions (20) we obtain that $\|\Phi(t, t_0, x_0)\| \le M$ for $t > t_0 \ge 0$ and $|x_0| < \delta$, which completes the proof of Theorem 9. ■

Remark 2. In the case when in $\mathbb{R}^n$ the Euclidean norm $|x|$ of the vectors $x \in \mathbb{R}^n$ is used, conditions (20) can be written down in the form

$$\int_\theta^\infty \mu(f_k(s, x(s;\, t_0, x_0))\, ds < \infty \qquad \text{and} \qquad \prod_{\theta < t_k < \infty} \|E + I_k'(0)\| < \infty$$

where

$$\mu(A) = \lim_{h \to 0^+} (1/h)[\|E + hA\| - 1]$$

is Lozinskii's "logarithmic norm" of the $n \times n$ matrix A.

Theorem 10. Let the following conditions be fulfilled:

1. Conditions (A) hold.
2. The zero solution of system (1) is uniformly Lipschitz stable.
3. $|\Phi(t, s, z)g(s, z)| \le \gamma(s)|z|$ for $t \ge s > t_0 \ge 0$, $z \in \mathbb{R}^n$, where Φ is the fundamental matrix of (3) and the function $g\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies conditions A2 and A3.

4. $|P_k(y)| \le \beta_k |y|$ for $y \in \mathbb{R}^n$, $k = 1, 2, \ldots$, where $P_k: \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition A4 and $\beta_k \ge 0$, $k = 1, 2, \ldots$, are constants.

5. $\|\Phi(t, t_k, y + I_k(y) + sP_k(y)\| \le \alpha_k$ for $t > t_0 \ge 0$, $y \in \mathbb{R}^n$, $0 \le s \le 1$, and $k = 1, 2, \ldots$, where $\alpha_k \ge 0$, $k = 1, 2, \ldots$, are constants.

6. $\int_\theta^\infty \gamma(t)\, dt < \infty$ and $\prod_{\theta < t_k < \infty} (1 + \alpha_k \beta_k) < \infty$ for $\theta > t_0 \ge 0$.

Then the zero solution of the perturbed impulsive system

$$\begin{aligned} \dot{y} &= f(t, y) + g(t, y), \qquad t \neq t_k \\ \Delta y|_{t=t_k} &= I_k(y) + P_k(y) \end{aligned} \tag{21}$$

is uniformly Lipschitz stable.

Proof. Using Alexeyev's variation of parameter nonlinear formula for impulsive differential equations (Simeonov and Bainov, 1987, p. 268), for the solution $y(t) = y(t; t_0, x_0)$ of (21) we obtain

$$\begin{aligned} y(t; t_0, x_0) = x(t; t_0, x_0) &+ \int_{t_0}^{t} \Phi(t, s, y(s)) g(s, y(s))\, ds \\ &+ \sum_{t_0 < t_k < t} \int_0^1 \Phi(t, t_k, y(t_k) + I_k(y(t_k)) \\ &+ sP_k(y(t_k)))\, ds \cdot P_k(y(t_k)) \end{aligned} \tag{22}$$

From condition 2 of Theorem 10 it follows that there exist constants $\alpha > 0$ and $\delta > 0$ such that

$$|x(t; t_0, x_0)| \le \alpha |x_0| \qquad \text{for} \quad t > t_0 \ge 0 \quad \text{and} \quad |x_0| < \delta \tag{23}$$

From conditions 3–5 of Theorem 10 and from (22) and (23) it follows that

$$|y(t)| = |y(t; t_0, x_0)| \le \alpha |x_0| + \int_{t_0}^{t} \gamma(s) |y(s)|\, ds + \sum_{t_0 < t_k < t} \alpha_k \beta_k |y(t_k)|$$

from which, using the impulsive integral inequality (Samoilenko and Perestyuk, 1987, Lemma 2.1), we obtain

$$|y(t; t_0, x_0| \le \alpha |x_0| \prod_{t_0 < t_k < t} (1 + \alpha_k \beta_k) \exp\left[\int_{t_0}^{t} \gamma(s)\, ds\right]$$

Then from condition 6 of Theorem 10 it follows that $|y(t; t_0, x_0)| \le \alpha |x_0|$ for $|x_0| < \delta$ and $t > t_0 \ge 0$.

Theorem 10 is proved. ■

Corollary 3. Let the following conditions be fulfilled:

1. Condition (B) holds.
2. The zero solution of system (6) is uniformly Lipschitz stable.

3. The function $g: J\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies conditions A2 and A3 and

$$|g(t,y)|\le\gamma(t)|y| \qquad \text{for} \quad y\in\mathbb{R}^n, \quad t>t_0\ge 0$$

where $\int_\theta^\infty \gamma(s)\,ds<\infty$ for $\theta>t_0\ge 0$.

4. The functions $P_k: \mathbb{R}^n\to\mathbb{R}^n$ satisfy condition A4 and $|P_k(y)|\le\beta_k|y|$, where $0\le\beta_k=\text{const}$ and $\prod_{\theta<t_k<\infty}(1+\alpha\beta_k)<\infty$ for $\theta>t_0\ge 0$ and $\alpha>0$.

Then the zero solution of the system

$$\dot y=A(t)y+g(t,y), \qquad t\ne t_k$$
$$\Delta y|_{t=t_k}=B_k y+P_k(y)$$

is uniformly Lipschitz stable.

In the subsequent theorems some relations among the notions of Lipschitz stability introduced by Definition 1 are considered.

Theorem 11. Let conditions (A) hold and let the zero solution of system (1) be uniformly Lipschitz stable in variations. Then the zero solution of (1) is uniformly Lipschitz stable.

Proof. From Lakshmikantham *et al.* (1989, Theorem 2.4.1) it follows that for $t\ne t_k$, $k=1,2,\dots$, we have

$$\frac{\partial}{\partial s}x(t;t_0,sx_0)=\Phi(t,t_0,x_0)x_0 \tag{24}$$

where Φ is the fundamental matrix of system (3).

Integrating (24) from 0 to 1, we obtain

$$x(t;t_0,x_0)=\left[\int_0^1\Phi(t,t_0,sx_0)\,ds\right]x_0, \qquad t>t_0\ge 0, \quad t\ne t_k, \quad k=1,2,\dots$$

From the condition of Theorem 11 it follows that there exist constants $M>0$ and $\delta>0$ such that

$$\|\Phi(t,t_0,y_0)\|\le M \qquad \text{for} \quad |y_0|<\delta \quad \text{and} \quad t>t_0\ge 0$$

and since $|sx_0|=s|x_0|\le|x_0|$, then for $t>t_0\ge 0$, $t\ne t_k$, $k=1,2,\dots$, and $|x_0|<\delta$ we have $\|\Phi(t,t_0,sx_0)\|\le M$.

Hence

$$|x(t;t_0,x_0)|\le\left[\int_0^1\|\Phi(t,t_0,sx_0)\|\,ds\right]|x_0|\le M|x_0|$$

for $t>t_0\ge 0$, $t\ne t_k$, $k=1,2,\dots$, and $|x_0|<\delta$.

From the continuity from the left of $x(t; t_0, x_0)$ at the points t_k, $k=1,2,\dots$, it follows that $|x(t_k; t_0, x_0)| \le M|x_0|$, $k=1,2,\dots$, with which Theorem 11 is proved. ■

Theorem 12. Let conditions (A) hold and let the zero solution of (1) be uniformly Lipschitz stable. Then the zero solution of (2) is uniformly Lipschitz stable.

Proof. Since the zero solution of (1) is uniformly Lipschitz stable, then there exist constants $M>0$ and $\delta>0$ such that $|x(t; t_0, x_0)| \le M|x_0|$ for $t>t_0\ge 0$ and $|x_0|<\delta$.

Let $x_0=(x_{01},\dots,x_{0j-1},x_{0j},x_{0j+1},\dots,x_{0n})$, $j=1,2,\dots,n$, and $|x_0|\le h<\delta$. Since $x(t; t_0, 0)=0$ for $t>t_0\ge 0$, $t\ne t_k$, $k=1,2,\dots$, we have

$$\left|\frac{\partial}{\partial x_{0j}}(x(t; t_0, 0))\right| = \left|\lim_{h\to 0}(1/h)[x(t; t_0, x_0)-x(t; t_0, 0)]\right|$$
$$\le \lim_{h\to 0} M|x_0|/h \le M$$

Hence

$$\|\Psi(t, t_0)\| = \|\Phi(t, t_0, 0)\| = \left\|\frac{\partial}{\partial x_0}(x(t; t_0, x_0))\right\| \le M$$

for $t>t_0\ge 0$, $t\ne t_k$, $k=1,2,\dots$. From the continuity from the left of $\Psi(t, t_0)$ at the points t_k, $k=1,2,\dots$, it follows that $\|\Psi(t_k, t_0)\|\le M$, $k=1,2,\dots$.

Hence the zero solution of (2) is uniformly stable (Samoilenko and Perestyuk, 1987, Theorem 8.1). From Theorem 1 it follows that the zero solution of (2) is uniformly Lipschitz stable. ■

Theorem 13. Let the following conditions hold:

1. Conditions (A) are satisfied.
2. The zero solution of (2) is uniformly Lipschitz stable.
3. For $\theta>t_0\ge 0$ and $|x_0|<\delta$ $(\delta>0)$

$$\int_\theta^\infty \|f_x(s, x(s; t_0, x_0))-f_x(s, 0)\|\, ds<\infty \tag{25}$$

$$\prod_{\theta<t_k<\infty} (1+K\|I_k'(x(t_k; t_0, x_0))-I_k'(0)\|)<\infty, \qquad K>0 \tag{26}$$

Then the zero solution of (1) is uniformly Lipschitz stable in variations.

Proof. Let $z(t)=z(t; t_0, z_0)$ be a solution of system (3) for which $z(t_0+0)=z_0$. Then system (3) can be written down in the form

$$\dot z = f_x(t, 0)z+[f_x(t, x(t; t_0, x_0))-f_x(t, 0)]z, \qquad t\ne t_k$$
$$\Delta z|_{t=t_k} = I_k'(0)z+[I_k'(x(t_k; t_0, x_0))-I_k'(0)]z$$
$$z(t_0+0)=z_0$$

From the variation of parameters formula for impulsive differential equations (Simeonov and Bainov, 1987, p. 266) we obtain

$$\begin{aligned} z(t;\, t_0,\, z_0) &= \Psi(t,\, t_0+0)z_0 \\ &\quad + \int_{t_0}^{t} \Psi(t,\, s)[f_x(s,\, x(s;\, t_0,\, x_0)) - f_x(s,\, 0)]z(s;\, t_0,\, z_0)\, ds \\ &\quad + \sum_{t_0<t_k<t} \Psi(t,\, t_k+0)[I_k'(x(t_k;\, t_0,\, x_0)) - I_k'(0)]z(t_k;\, t_0,\, z_0) \end{aligned} \tag{27}$$

From condition 2 of Theorem 13 and from Theorem 1 it follows that there exists a constant $K>0$ such that

$$\|\Psi(t,\, t_0)\| \leq K \qquad \text{for} \quad t>t_0\geq 0 \tag{28}$$

From (27) and (28) it follows that

$$\begin{aligned} &|z(t;\, t_0,\, z_0)| \\ &\leq K|z_0| + K \int_{t_0}^{t} \|f_x(s,\, x(s;\, t_0,\, x_0)) - f_x(s,\, 0)\|\, |z(s;\, t_0,\, z_0)|\, ds \\ &+ K \sum_{t_0<t_k<t} \|I_k'(x(t_k;\, t_0,\, x_0)) - I_k'(0)\|\, |z(t_k;\, t_0,\, z_0)| \end{aligned}$$

Applying the impulsive integral inequality (Samoilenko and Perestyuk, 1987, Lemma 2.1), we obtain

$$\begin{aligned} &|z(t;\, t_0,\, z_0| \\ &\leq K|z_0| \prod_{t_0<t_k<t} [1+K\|I_k'(x(t_k;\, t_0,\, x_0)) - I_k'(0)\| \\ &\times \exp\left[K \int_{t_0}^{t} \|f_x(s,\, x(s;\, t_0,\, x_0)) - f_x(s,\, 0)\|\, ds\right] \end{aligned}$$

Then from conditions (25) and (26) it follows that $|z(t;\, t_0,\, z_0| \leq M|z_0|$ for $t>t_0\geq 0$.

Hence

$$\|\Phi(t,\, t_0,\, x_0)\| = \sup_{|z_0|\leq 1} |\Phi(t,\, t_0,\, x_0)z_0| = \sup_{|z_0|\leq 1} |z(t;\, t_0,\, z_0)| \leq \sup_{|z_0|\leq 1} M|z_0| \leq M$$

Consequently, the zero solution of system (1) is uniformly Lipschitz stable in variations. ■

From Theorems 12 and 13 we obtain the following corollary.

Corollary 4. Let conditions 1 and 3 of Theorem 13 hold and let the zero solution of system (1) be uniformly Lipschitz stable. Then the zero solution of (1) is uniformly Lipschitz stable in variations.

From Theorems 11 and 13 we obtain the following corollary.

Corollary 5. If the conditions of Theorem 13 hold, then the zero solution of system (1) is uniformly Lipschitz stable.

4. EXAMPLES

We shall illustrate the results obtained by some examples.

Example 1. Consider the linear impulsive system

$$\dot{x} = Ax, \quad t \neq t_k; \qquad \Delta x|_{t=t_k} = B_k x; \qquad x(t_0+0) = x_0 \tag{29}$$

where A and B_k, $k = 1, 2, \ldots$, are constant $n \times n$ matrices. By straightforward calculations we establish that $[x, Ax]_+ \leq \mu(A)|x|$.

Let $\|E + B_k\| \leq d_k$, $k = 1, 2, \ldots$, and let condition A1 hold. Consider the scalar impulsive differential equation

$$\dot{u} = \mu(A)u, \quad t \neq t_k; \qquad \Delta u|_{t=t_k} = (d_k - 1)u; \qquad u(t_0+0) = u_0 \geq 0 \tag{30}$$

whose solution is

$$u(t; t_0, u_0) = u_0\left(\prod_{t_0 < t_k < t} d_k\right) \exp[\mu(A)(t - t_0)]$$

If we suppose that $\prod_{k=1}^{\infty} d_k$ is convergent and $\mu(A) \leq 0$, then the zero solution of (30) is globally uniformly Lipschitz stable. Then from Theorem 6 it follows that the zero solution of system (29) is globally Lipschitz stable.

Example 2. Consider the linear impulsive system (6) for which conditions A1 and (B) hold. If, moreover, the following conditions hold:

(a) $\limsup_{t\to\infty}\left[\int_{t_0}^{t} \mu(A(s))\, ds\right] < \infty$

(b) $\|E + B_k\| \leq d_k, \quad k = 1, 2, \ldots$

(c) $\prod_{k=1}^{\infty} d_k < \infty$

then the zero solution of the scalar impulsive differential equation

$$\dot{u} = \mu(A(t))u, \quad t \neq t_k; \qquad \Delta u|_{t=t_k} = (d_k - 1)u; \qquad u(t_0+0) = u_0 \geq 0$$

is globally uniformly Lipschitz stable. Then from Theorem 6 it follows that the zero solution of system (6) is globally uniformly Lipschitz stable.

Example 3. Consider the impulsive system of differential equations (1). Let conditions (A) hold as well as the following conditions:

(a) $[x, f(t, x)]_+ \leq p(t)F(|x|)$ for $(t, x) \in J \times S(\rho)$, where $p \in C[\mathbb{R}_+, \mathbb{R}_+]$ and $F \in \mathscr{K}$ ($\mathbb{R}_+ = [0, \infty)$).

(b) $|x + I_k(x)| \leq G_k(|x|)$ for $x \in S(\rho)$, $k = 1, 2, \ldots$, where $G_k\colon [0, \rho_0) \to [0, \rho)$ and $G_k \in \mathscr{K}$, $k = 1, 2, \ldots$.

(c) For any $\sigma \in (0, \rho_0]$ the following inequality holds:

$$\int_{t_k}^{t_{k-1}} p(s)\, ds + \int_{\sigma}^{G_k(\sigma)} \frac{ds}{F(s)} \leq 0, \qquad k = 1, 2, \ldots$$

Then the zero solution of the scalar impulsive differential equation

$$\dot{u} = p(t)F(u), \quad t \neq t_k; \qquad \Delta u|_{t=t_k} = G_k(u(t_k)) - u(t_k); \qquad u(t_0+0) = u_0 \geq 0$$

is uniformly Lipschitz stable. From Theorem 6 it follows that the zero solution of system (1) is uniformly Lipschitz stable.

Example 4. Consider the impulsive system

$$\dot{x} = y, \quad \dot{y} = -x^{2n+1}, \quad t \neq t_k; \qquad \Delta x|_{t=t_k} = \alpha_k x; \qquad \Delta y|_{t=t_k} = \alpha_k y \tag{31}$$

where $n \geq 1$ is an integer and $\prod_{k=1}^{\infty} (1 + \alpha_k) < \infty$. The solution $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ of system (31) satisfies the relation

$$x^{2n+2}/(n+1) + y^2 = [x_0^{2n+2}/(n+1) + y_0^2] \prod_{t_0 < t_k < t} (1 + \alpha_k)^2$$

Hence the zero solution of system (31) is uniformly stable.

Consider the variational system of (31) corresponding to its zero solution:

$$\dot{u} = v, \quad \dot{v} = 0, \quad t \neq t_k; \qquad \Delta u|_{t=t_k} = \alpha_k u; \qquad \Delta v|_{t=t_k} = \alpha_k v \tag{32}$$

whose solution is

$$u(t; t_0, u_0, v_0) = \left[\prod_{t_0 < t_k < t} (1 + \alpha_k)\right][u_0 + v_0(t - t_0)]$$

$$v(t; t_0, u_0, v_0) = v_0 \prod_{t_0 < t_k < t} (1 + \alpha_k)$$

Hence the zero solution of system (32) is unstable. From Theorem 12 it follows that the zero solution of (31) is not uniformly Lipschitz stable.

Example 5. Consider the impulsive system

$$\begin{aligned} \dot{x} &= n(t)y + m(t)x(x^2 + y^2), \qquad t \neq t_k \\ \dot{y} &= -n(t)x + m(t)y(x^2 + y^2), \qquad t \neq t_k \\ \Delta x|_{t=t_k} &= \alpha_k x, \qquad \Delta y|_{t=t_k} = \alpha_k y \end{aligned} \tag{33}$$

where the functions $m, n\colon \mathbb{R}_+ \to \mathbb{R}$ are piecewise continuous with points of discontinuity of the first kind t_k at which they are continuous from the left.

A straightforward verification yields that if $m(t) > 0$, $\int_0^\infty m(t)\,dt = \infty$, $t_k - t_{k-1} \geq \theta > 0$ $(k = 1, 2, \ldots)$ and $\prod_{k=1}^{\infty} (1 + \alpha_k) < \infty$, then the zero solution of system (33) is unstable.

Consider the variational system of (33) corresponding to its zero solution:

$$\dot{u} = n(t)v, \qquad \dot{v} = -n(t)u, \qquad t \neq t_k$$
$$\Delta u\big|_{t=t_k} = \alpha_k u, \qquad \Delta v\big|_{t=t_k} = \alpha_k v \tag{34}$$

It is clear that

$$\rho(t;\, t_0, \rho_0) = \rho_0 \prod_{t_0 < t_k < t} (1 + \alpha_k)$$

where

$$\rho(t;\, t_0, \rho_0) = u^2(t;\, t_0, u_0, v_0) + v^2(t;\, t_0, u_0, v_0), \qquad \rho_0^2 = u_0^2 + v_0^2$$

Hence the zero solution of system (34) is uniformly Lipschitz stable. This shows that Theorem 12 is not invertible.

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